The State Sensitivity Analysis of the Front Wheel Steering Vehicle : In the Time Domain

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In this paper, a sensitivity analysis of a front wheel steering vehicle in the time domain is considered. For this study, a two degree-of-freedom bicycle model is used. This model displays the simplest lateral dynamic effect and is useful for understanding of the dynamic characteristics and control aspects of the target system. The side slip angle and yaw rate are selected as the system state variables. Vehicle mass, inertia, cornering stiffness, and wheel base are taken as design variables. Sensitivity analyses are performed by the direct differentiation method, which is an efficient tool for error control in numerical integration. This research proposes a basis for re-design and new-design of a vehicle by checking variations in the state variable with respect to changes in the design variable. Finally, dominant design variables are suggested through simulations.

Key Words: Sensitivity Analysis, Front Wheel Steering Vehicle, Time Domain, Lateral Vehicle Dynamics, Direct Differentiation Method

Nomenclature

- m : Vehicle mass
- I : Moment of inertia of the vehicle in yaw direction
- V : Vehicle speed
- K_f : Cornering stiffness of the front wheel
- K_r : Cornering stiffness of the rear wheel
- l_f : Distance from c.g. to front wheel center
- l_r : Distance from c.g. to rear wheel center
- δ_f : Front wheel steering angle

1. Introduction

Sensitivity analysis is an efficient tool for checking the variations in the design variables with respect to the system state variables (Deif, 1986). State variables represent dynamic characteristics of the system. Otherwise, design variables are specified by the design criteria. So, the characteristics of the state variables are governed by the initial value of the design variables. When of re-designing a system, sensitivity information can be used as a design basis (Vanderplaats, 1984; Arora, 1989). Therefore, sensitivity analysis is an important tool for the designer.

Sensitivity analysis can be classified according to static, kinematic, and dynamic with respect to the system state. Static sensitivity analysis is used to check the variation of the state variable with respect to the design variable in the static state. Particularly, this can be done in the structural design phase. Kinematic sensitivity analysis can be done in the kinematic state of the mechanism (Haug and Sohoni, 1984). Dynamic sensitivity analysis is utilized for variation evaluation for mechanism in the dynamic state (Haug, Mani, and Krishnaswami, 1984). Also, for the evaluation of the steady state characteristics of the target system, a frequency domain analysis is necessary (Jang and Han, 1997a, 1997b). In this paper, only

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dynamic sensitivity analysis is performed for the lateral vehicle dynamic system.

The methods for sensitivity analysis which are conventionally used are the adjoint variable method and the direct differentiation method (Haug, Mani, and Krishnaswami, 1984). There are two integration processes for dynamic analysis (forward) and sensitivity analysis (backward) in the adjoint variable method. It is useful for treatment of systems which have a large number of design variables. This method also has some demerits. First, the error control for the backward integration process is not satisfactory. Secondly, the formulation process for sensitivity analysis is relatively difficult (Krishnaswami, Wehage, and Haug, 1983; Chang and Nikravesh, 1985). Otherwise, the direct differentiation method is a robust algorithm to numerical error in the numerical integration process. It requires only forward integration for dynamic and sensitivity analysis. Also, derivation of the sensitivity functions is very straightforward. So, in this research, the later method is implemented and used for sensitivity analysis of vehicle lateral dynamic problems.

Lateral vehicle dynamics is concerned with steering maneuvers when the automobile is driving with variable forward speeds. Generally, steering maneuver of the vehicle is carried out by the steering wheel input of the driver. The vehicle is then steered to the target path of the driver's desire. This is one of the important maneuver for maneuverability of the vehicle (Whitehead, 1988). Front wheel steering vehicle is commonly used in most of today's passenger cars.

In order to study the dynamic characteristics and state sensitivity with respect to the design variables, some type of vehicle model is required. It is also necessary to implement some control logic for the electronic control unit. First, a simple bicycle vehicle model is widely used for checking the validity of the developed control logic. It is also used for evaluation of basic lateral dynamic response (Ellis, 1969; Gillespie, 1992; Shiotsuka, Nagamatsu, and Yoshida, 1993). This model is the simplest one for verification of some types of steering maneuver (Jansen and van Oosten, 1994). The second candidate is a lumped multi-degree of freedom vehicle model. It is only modeled by a mass, spring, and damper for the whole vehicle modeling. This is the intermediate form of the vehicle model for steering and suspension system effects. The last form of the vehicle model is a multi-body vehicle model which has mass, spring, damper, and mechanical joints. It is useful for examination of more detailed vehicle characteristics such as three dimensional information (Jang, Jeong, and Han, 1995). This is a final version of the simulation part for design error correction and for modification of the developed control logic. For easy derivation of the dynamic and state sensitivity analysis, a simple bicycle model will be used in this research. The first model is simple and shows the dominant lateral dynamic model for dynamic analysis, control, and state sensitivity analysis.

The objective of this research is a study of the state sensitivity information with respect to changes in the design variable. In this paper, a simple bicycle model is used for treatment of the lateral vehicle dynamics. Both dynamic and state sensitivity analysis are performed with the model. The direct differentiation method is used for sensitivity analysis. Numerical integration for the two analyses are simultaneously performed by the same integration routine. In order to elaborate efficiency and precision in numerical analysis, the Adams predictor-corrector with variable stepsize and variable order is adopted. Finally, the dominant design variables of the system are suggested via analysis, and trends for the re-design and new-design of the vehicle are proposed.

2. Vehicle Model

In this paper, the major concern is with the lateral vehicle dynamics in steering maneuvers. Although several vehicle models can be used for this purpose, a simple bicycle model is selected for the analytical derivation of the sensitivity equations with respect to design variables. A vehicle in steering maneuver is represented by Fig. 1.

Each of the parameters and design variables are listed in the nomenclature section. In this paper,

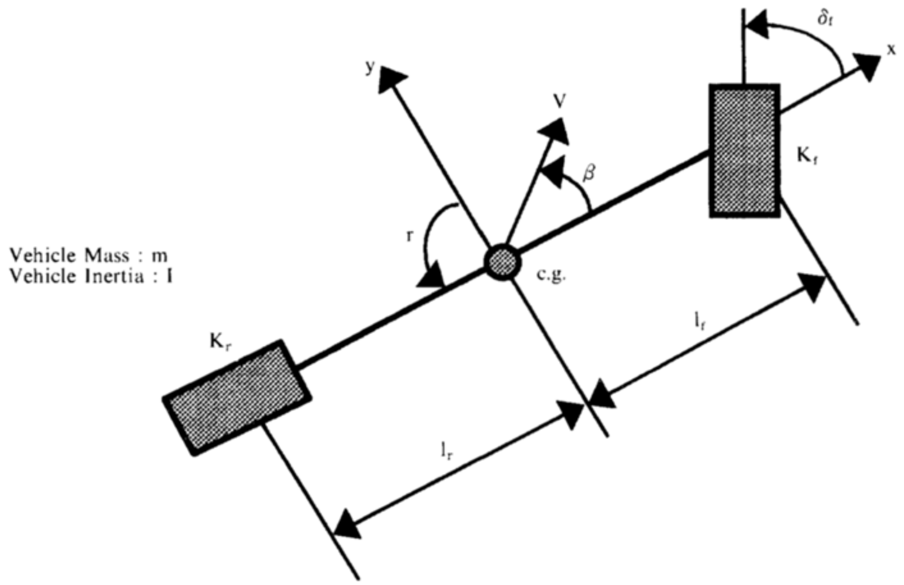

Fig. 1 A vehicle in steering maneuve

side slip angle and yaw rate are selected as the state variables of the system.

$$\underline{z}=[\beta \ \ r]^T \tag{1}$$

where, $\underline{z}$ is the state variable vector, β is the side slip angle, and r represents the yaw rate of the vehicle. A super script T denotes transpose of a vector or matrix. Design variables of this system are selected to be the following:

$$\begin{aligned}\underline{b}&=[b_1 \ b_2 \ b_3 \ b_4 \ b_5 \ b_6]^T\\ &=[m \ I \ K_f \ K_r \ l_f \ l_r]^T\end{aligned} \tag{2}$$

The governing equations of motion for the front wheel steering vehicle system can be expressed in the following form (Jang, 1995). This equation can be derived with the assumption that the side slip angle is very small during normal driving conditions.

$$\dot{\underline{z}}=\begin{bmatrix} -\dfrac{2(K_f+K_r)}{mV}\beta-[1+\dfrac{2}{mV^2} \\ (l_fK_f-l_rK_r)]r+\dfrac{2K_f}{mV}\delta_f \\ -\dfrac{2(l_fK_f-l_rK_r)}{I}\beta-\dfrac{2}{IV} \\ (l_f^2K_f+l_r^2K_r)r+\dfrac{2l_fK_f}{I}\delta_f \end{bmatrix} \tag{3}$$

This equation has the form of a first order ordinary differential equation. It is the dynamic equation for the vehicle lateral dynamics. Equation (3) can be represented by the following general form of the state equation:

$$\dot{\underline{z}}=\underline{f}(\underline{z}, \ \underline{b}, \ t) \tag{4}$$

where, $\dot{\underline{z}}$ is the time derivative of the state variable vector, $\underline{z}$, and t is time. From Eq. (4), the time derivative of the state variable can be represented by a function of the state variable vector, design variable vector, and time. It is differentiated with respect to the design variables for the state sensitivity analysis.

3. Sensitivity Formulation

In order to perform sensitivity analysis, state sensitivity equations must be derived with respect to the design variables at the initial stage. A general form of the first order differential sensitivity equations is

$$\frac{\partial \dot{\underline{z}}}{\partial \underline{b}}=\frac{\partial \underline{f}}{\partial \underline{z}}\cdot\frac{d\underline{z}}{d\underline{b}}+\frac{\partial \underline{f}}{\partial \underline{b}} \tag{5}$$

This equation can be written in the following compact matrix form:

$$\dot{\underline{z}}_b=\underline{f}_z\cdot\underline{z}_b+\underline{f}_b \tag{6}$$

where, $\underline{z}\in\mathbb{R}^n$, $\underline{b}\in\mathbb{R}^m$, $\underline{f}\in\mathbb{R}^n$, $\underline{z}_b\in\mathbb{R}^{n\times m}$, $\underline{f}_z\in\mathbb{R}$

$^{n\times n}$, and $\underline{f}_b \in \mathbb{R}^{n\times m}$. n is the number of state variables and m is the number of design variables. In this case, n is 2 and m is 6 from Eqs. (1) and (2). $\underline{z}_b$ is a state sensitivity matrix with respect to the design variables, and, $\dot{\underline{z}}_b$ is the time derivative of the state sensitivity matrix. In case of linear differential equations, $\underline{f}_z$ can be expressed by the system matrix A. From this idea, Eq. (6) can be re-written in the following form:

$$\dot{\underline{z}}_b = A \cdot \underline{z}_b + \underline{f}_b \tag{7}$$

The system matrix A is expressed as follows:

$$A = \begin{bmatrix} -\frac{2(K_f+K_r)}{mV} & -[1+\frac{2}{mV^2}(l_fK_f - l_rK_r)] \\ -\frac{2(l_fK_f - l_rK_r)}{I} & \frac{2}{IV}(l_f^2K_f + l_r^2K_r) \end{bmatrix} \tag{8}$$

Next, we must obtain the last terms, $\underline{f}_b$, for state sensitivity analysis with respect to the design variables. This term is expressed in following matrix form:

$$\underline{f}_b = [\underline{f}_{b1}, \underline{f}_{b2}, \underline{f}_{b3}, \underline{f}_{b4}, \underline{f}_{b5}, \underline{f}_{b6}]_{2\times 6} \tag{9}$$

Detailed derivations of the above are summarized in the Appendix.

In order to solve the dynamic and state sensitivity equations of the system, Eqs. (3) and (7) are simultaneously integrated. For efficiency and precision of the numerical analysis, a fairly robust integration routine must be used. In this research, the Adams predictor-corrector variable stepsize/variable order method is adopted. In this program, the order and stepsize of the differential equation solver is automatically changed depending on integration conditions.

4. Results of the Sensitivity Analysis

Various sensitivity analyses are performed about a nominal front wheel vehicle system in the time domain. The Vehicle data used in this paper is listed in Table 1. These data are general for conventional mid-size passenger car. From these data, we can determine whether the car has understeer characteristics and/or stability for the steering maneuver.

In this paper, we perform a step steering response assuming 60 degree steering wheel input, And a gear ratio of 15 between the steering wheel and front wheel. Sensitivity is evaluated at various vehicle speeds (20 km/h ~ 120 km/h). The effects of each of the design variables on the state variables are examined, and a dominant design variable is determined from the results.

Before sensitivity analysis, dynamic analysis of the side slip angle and yaw rate with respect to

Table 1 Vehicle data list

Design variables	Data	Dimension
m	1,300	kg
I	2,100	kg.m^2
K_f	40,000	N/rad
K_r	30,000	N/rad
l_f	1.01	m
l_r	1.65	m

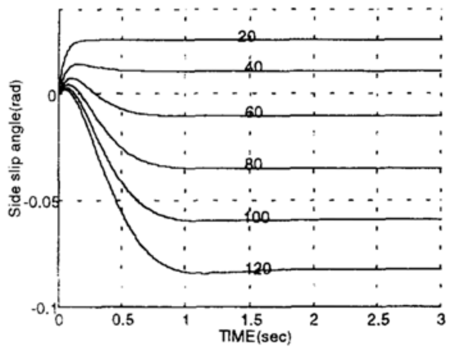

Fig. 2 Side slip angle with respect to vehicle speed (km/h)

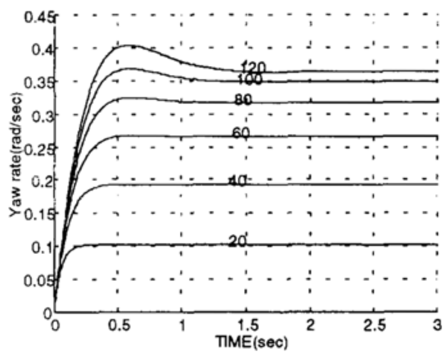

Fig. 3 Yaw rate with respect to vehicle speed (km/h)

vehicle speed is also examinid. The results of the side slip angle and yaw rate are represented in Figs. 1 and 2. From these figures, we know that the states of the vehicle are changed with vehicle forward speed. The values of side slip angle decrease as the vehicle speed is increased. On the other hand, the values of the yaw rate increase as the vehicle speed increased. The results are quite reasonable compared with previous works (Whitehead, 1988; Jang, 1995)

4.1 The effects of the vehicle mass

The effects of vehicle mass on state sensitivity are considered in the time domain. Figure 4 shows the results of the side slip angle sensitivity with respect to changes in the vehicle mass. It shows that the sensitivity of the side slip angle for vehicle mass increases in the negative direction. As the vehicle speed increases, the settling time to steady state value is much longer than at low speeds. Figure 5 is the result of the yaw rate sensitivity analysis with respect to changes in vehicle mass. The sensitivity value of the yaw rate for vehicle mass negatively increases with the vehicles forward speed. Like the previous case, the settling time for the value increases. Also, there is non-oscillatory behavior in the time domain.

4.2 The effects of the vehicle inertia

The effects of vehicle inertia in the yaw direction to state sensitivity are considered in the time domain. Figure 6 is the result of the side slip angle sensitivity analysis with respect to changes in vehicle inertia. We can see that overshoot and oscillatory behavior occurs as the speed is increased. A zero point of the sensitivity value is detected in this case. Since side slip angle has maximum value at this point, it must be carefully examined. Usually, one can say that a vehicle is

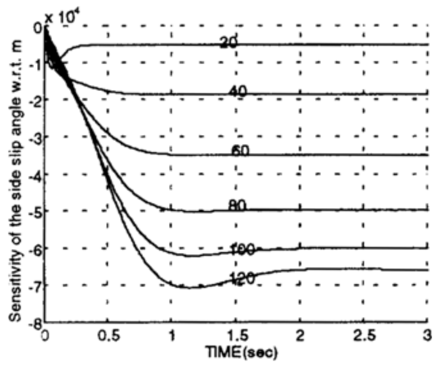

Fig. 4 Side slip angle sensitivity with respect to the change of m

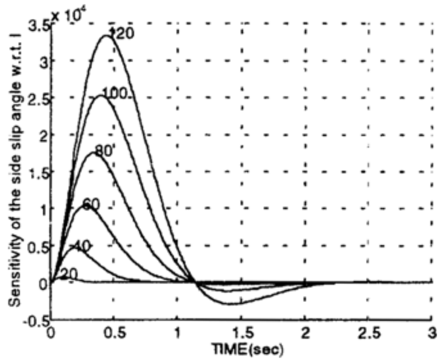

Fig. 6 Side slip angle sensitivity with respect to the change of I

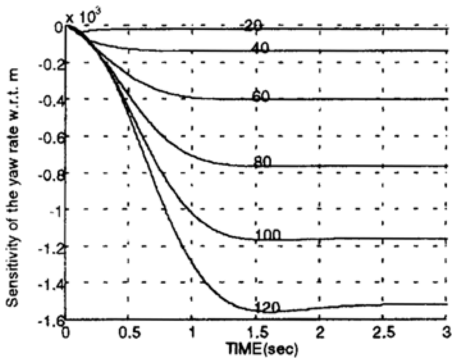

Fig. 5 Yaw rate sensitivity with respect to the change of m

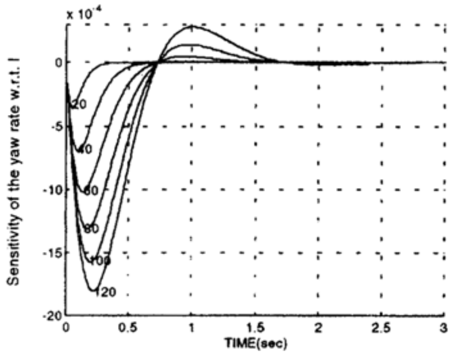

Fig. 7 Yaw rate sensitivity with respect to the change of I

dynamically very uncertain at such a carfiguration. The maximum sensitivity value positively increases with an increase in vehicle speed. Figure 7 shows the yaw rate sensitivity with respect to changes in the vehicle inertia. The sensitivity of the yaw rate for inertia has the same oscillatory characteristics as Fig. 6, but, the maximum sensitivity value negatively increases as the vehicle speed is increased. The zero point of the sensitivity variable occurrs at about 0.7 Sec. Contrary to the previous case, the yaw rate value has minimum value at this point. This inverse behavior for side slip angle and yaw rate change for the vehicle inertia needs to be carefully considered.

4.3 The effects of the cornering stiffness of the front wheel

The effects of cornering stiffness of the front wheel an state sensitivity are considered in the time domain. Figure 8 represents the side slip angle sensitivity with respect to changes in the cornering stiffness of the front wheel. In the range of 0～40 km/h vehicle speed, the sensitivity values have positive quantities. Above this speed, the sensitivity value negatively increases as the vehicle speed increases. All sensitivity values reach a zero point before 0.5 sec. The yaw rate sensitivity with respect to changes in the cornering stiffness of the front wheel is shown in Fig. 9. In this case, all sensitivity values have positive ralues, any zero points of the sensitivity values are not detected. The sensitivity value positively increases as the vehicle speed is increased.

4.4 The effects of the cornering stiffness of the rear wheel

The effects of the cornering stiffness of the rear wheel on state sensitivity are considered in the time domain. Figure 10 shows the results of the side slip angle sensitivity with respect to changes

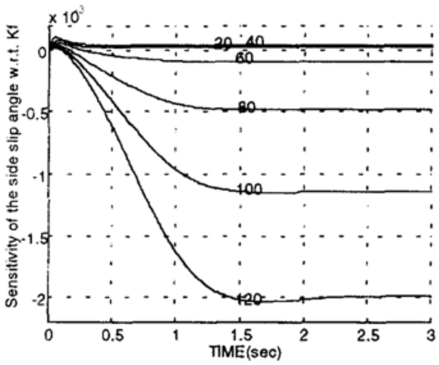

Fig. 8 Side slip angle sensitivity with respect to the change of K_f

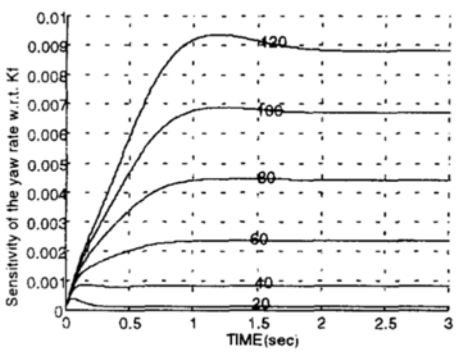

Fig. 9 Yaw rate sensitivity with respect to the change of K_f

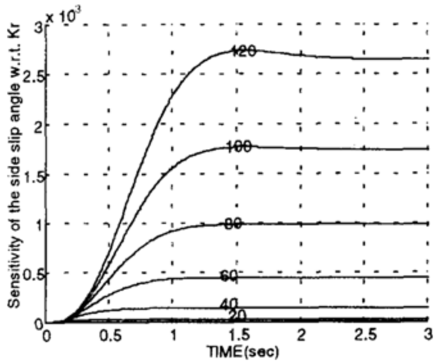

Fig. 10 Side slip angle sensitivity with respect to the change of K_r

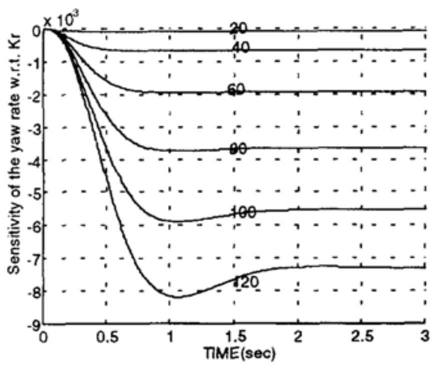

Fig. 11 Yaw rate sensitivity with respect to the change of K_r

in the cornering stiffness of the rear wheel. The sensitivity values of side slip angle for cornering stiffness of the front wheel positively increase as the vehicle speed is increased. Any zero points of the sensitivity value do not exist in this case. At about 120 km/h an overshoot in the sensitivity value is detected. The result of the yaw rate sensitivity with respect to changes in the cornering stiffness of the rear wheel are shown in Fig. 11. In this case, the sensitivity value negatively increases as the vehicle speed is increased. Like the previous case, overshoots are detected at high speeds above 100 km/h. At this point the sensitivity value has a maximum. So, the vehicle response to changes in the cornering stiffness of the rear wheel may be uncertain. The results of Figs. 10 and 11 for the rear wheel show opposite trends to those of Figs. 8 and 9 with respect to the cornering stiffness of the front wheel.

4.5 The effects of the distance from c.g. to front wheel center

The effects of the distance from the center of gravity to the front wheel center on state sensitivity are considered in the time domain. Figure 12 shows the results of the side slip angle sensitivity with respect to changes in the distance from the center of gravity to the front wheel center. In this case, the values negatively increase as the vehicle speed increased. The results of the yaw rate sensitivity analysis with respect to changes in the distance from the center of gravity to front wheel center are shown in Fig. 13. The sensitivity values positively increase as the vehicle speed is increased. This trend is quite different from the previous case. The overshoot for the sensitivity value is detected in the high speed range (above 100 km/h).

4.6 The effects of the distance from c.g. to rear wheel center

The effects of the distance from the center of gravity to the rear wheel center on state sensitivity are considered in the time domain. The results of the side slip angle sensitivity analysis with respect to changes in the distance from center of gravity to the rear wheel center are shown in Fig. 14. In this case, the values positively increase as the vehicle speed increased. Any zero points of the sensitivity value are not detected. Figure 15 is the result of the yaw rate sensitivity with respect to changes in the distance from the center of gravity

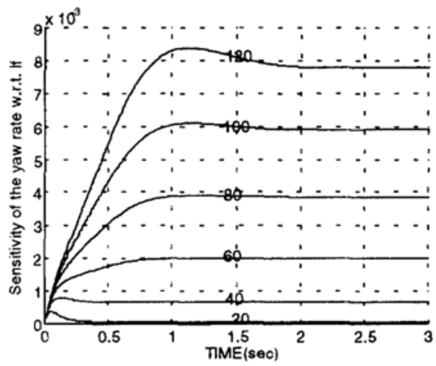

Fig. 13 Yaw rate sensitivity with respect to the change of l_f

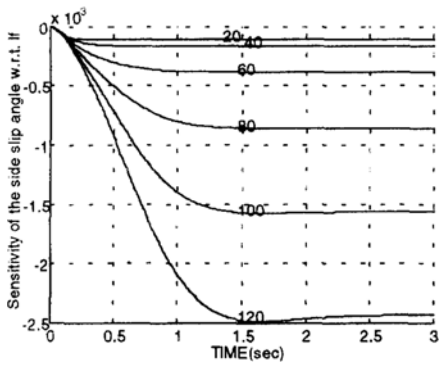

Fig. 12 Side slip angle sensitivity with respect to the change of l_f

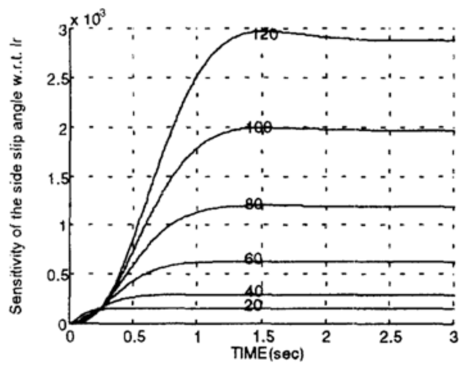

Fig. 14 Side slip angle sensitivity with respect to the change of l_r

to the rear wheel center. The sensitivity values negatively increase as the vehicle speed is increased. As in the previous case, no zero points occurr. The results of Figs. 14 and 15 show opposite trends to those of Figs. 12 and 13.

4.7 Dominant design variable check

The dominant design variables can be examined based on state sensitivity analyses. In order to check the dominant design variable, a normalization process is required for dimension matching. In this research, a 1% perturbation is performed for this process. The rank of the design variables very as the vehicle speed is increased. In this paper, only two vehicle speeds (40 km/h and 80 km/h) are considered.

In case of 40 km/h vehicle speed, the results are grouped in Figs. 16 and 17. From Fig. 16, we see that a dominant design variable for side slip angle is l_r, followed in turn by m, l_f, K_r, K_f, and I. In spite of K_f and I having minor effects on the side slip angle, these variables must be considered as significant terms; since these terms have oscillatory sensitivity characteristics at the initial stage (0 ~0.5 sec), we must carefully consider this point. Figure 17 implies that l_r is the dominant

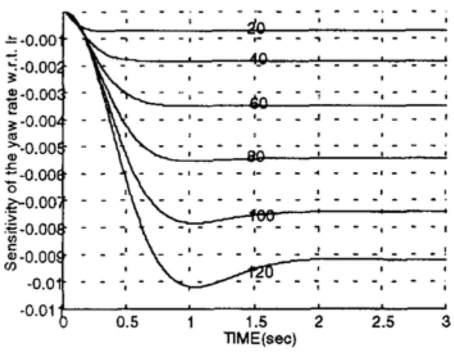

Fig. 15 Yaw rate sensitivity with respect to the change of l_r

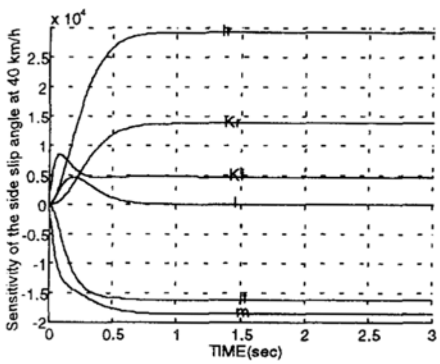

Fig. 16 Side slip angle sensitivity with respect to each design variables at 40km/h

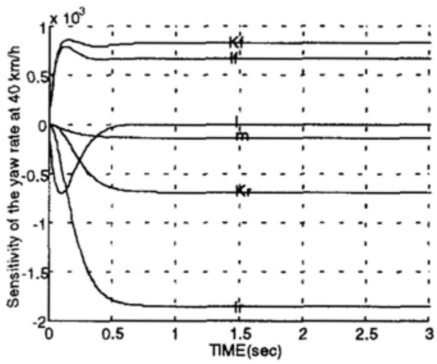

Fig. 17 Yaw rate sensitivity with respect to each design variables at 40km/h

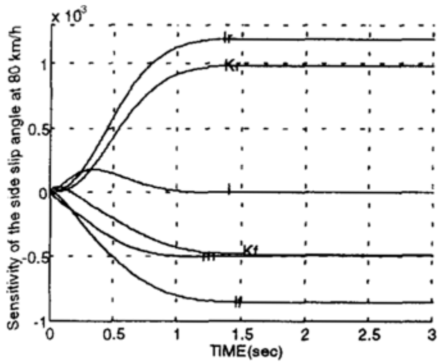

Fig. 18 Side slip angle sensitivity with respect to each design variables at 80km/h

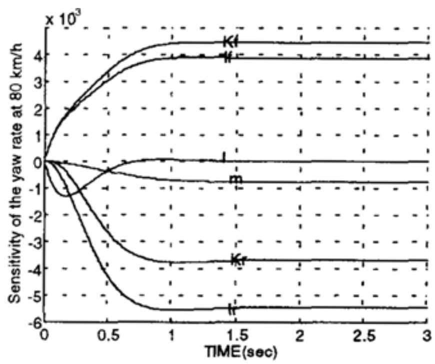

Fig. 19 Yaw rate sensitivity with respect to each design variables at 80km/h

design variable for yaw rate, followed in turn by K_f, K_r, l_f, m, and I. Like the previous case, I has the lowest rank, but it is separately treated as a special design variable due to its oscillatory characteristics.

In the case of 80 km/h vehicle speed, the results are shown in Figure 18 and 19. From Fig. 18, like the previous case, we see that a dominant design variable for the side slip angle is l_r, followed in turn by K_r, l_f, K_f, m, and I. From Fig. 19, l_r is the dominant design variable for the yaw rate, followed by K_f, l_f, K_r, m, and I.

Note that the sensitivity variables for the design variables change as the vehicle speeds up and down. This change point of the vehicle response is detected at about 40~60 km/h. This could be used as a basis for the development of new-robust control logic for external disturbances. Also, this may be used by a vehicle designer in the re-design or new-design stage of a vehicle.

5. Conclusions

In this study, a sensitivity analysis of the front wheel steering vehicle system is considered in the time domain. This analysis is performed by the direct differentiation method, which is an efficient tool for error control in numerical integration. The dynamic and state sensitivity equations are simultaneously integrated by the same integration routine. From various sensitivity analyses, useful sensitivity information is evaluated at various vehicle speeds with respect to the design variables. First, the effects of each of the design variables on the state variables are examined. Next, a dominant design variable is checked by comparison of the results. In these results, zero points of the state sensitivity variable-which is a critical point for vehicle design and control-are suggested. These points must be carefully considered as a design base. Also, we see that the effects of the distance from the center of gravity to the rear wheel center and to the front wheel center on state sensitivity are the most sensitive design variables. Also, cornering stiffness of the front wheel and rear wheel has the second highest rank among the sensitivity values. Because severe problems may occur at these points, e.s., loss of control of the vehicle, the zero points of the state sensitivity results must be outside the nominal driving conditions. In the case of newly designing a vehicle, a modification of the vehicle wheel base must be carefully treated to obtain a reasonable vehicle design. A sensitivity analysis of the rear wheel steering and four wheel steering vehicle will be studied for developing high performance vehicle concept in the near future.

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Appendix

Sensitivity functions for RHS terms of the Eq. (3) are derived as follows. Total six sensitivity equations for each design variables are obtained from partial derivative operation with respect to the design variables.

① For $b_1 = m$

$$\underline{f}_{b_1} = \begin{bmatrix} \frac{2(K_f+K_r)}{m^2 V}\beta + \frac{2}{m^2 V^2}(l_f K_f - l_r K_r)\, r - \frac{2K_f}{m^2 V}\delta_f \\ 0 \end{bmatrix} \tag{A.1}$$

② For $b_2 = I$

$$\underline{f}_{b_2} = \begin{bmatrix} 0 \\ \frac{2(l_f K_f - l_r K_r)}{I^2}\beta + \frac{2}{I^2 V}(l_f^2 K_f + l_r^2 K_r)\, r - \frac{2 l_f K_f}{I^2}\delta_f \end{bmatrix} \tag{A.2}$$

③ For $b_3 = K_f$

$$\underline{f}_{b_3} = \begin{bmatrix} -\frac{2}{mV}\beta - \frac{2 l_f}{m V^2} r + \frac{2}{mV}\delta_f \\ -\frac{2 l_f}{I}\beta - \frac{2 l_f^2}{I V} r + \frac{2 l_f}{I}\delta_f \end{bmatrix} \tag{A.3}$$

④ For $b_4 = K_r$

$$\underline{f}_{b_4} = \begin{bmatrix} -\frac{2}{mV}\beta + \frac{2 l_r}{m V^2} r \\ \frac{2 l_r}{I}\beta - \frac{2 l_r^2}{I V} r \end{bmatrix} \tag{A.4}$$

⑤ For $b_5 = l_f$

$$\underline{f}_{b_5} = \begin{bmatrix} -\frac{2K_f}{m V^2} r \\ -\frac{2K_f}{I}\beta - \frac{2 l_f K_f}{I V} r + \frac{2K_f}{I}\delta_f \end{bmatrix} \tag{A.5}$$

⑥ For $b_6 = l_r$

$$\underline{f}_{b_6} = \begin{bmatrix} \frac{2K_r}{m V^2} r \\ \frac{2K_r}{I}\beta - \frac{4 l_r K_r}{I V} r \end{bmatrix} \tag{A.6}$$